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Generalized Model for Λ -Dark Energy

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Einstein field equations under spherically symmetric space-times are considered here in connection to dark energy investigation. A set of solutions are obtained for a kinematical Λ model, viz., $\Lambda \sim (\dot{a}/a)^2$ without assuming any *a priori* value for the curvature constant and the equation of state parameter ω . Some interesting results, such as the nature of cosmic density Ω and deceleration parameter q , have been obtained with the consideration of two-fluid structure instead of usual uni-fluid cosmological model.

Keywords: dark energy, Λ -model, curvature constant.

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1. Introduction

Present-day cosmological research hovers around the investigation of *dark energy*, an exotic type of entity responsible for generating acceleration in expanding Universe. In fact, various recent observational results^{1,2,3,4,5,6,7,8} suggest that the Universe is expanding with an acceleration alone while some other works^{9,10,11} indicate that the acceleration is a phenomenon of recent past and was preceded by a phase of deceleration.

Now, the exact nature of dark energy being still unknown, its investigation are going on along various paths. Phenomenological models are also contenders in this dark energy investigation. Although these type of phenomenological models do not originate from any underlying quantum field theory, yet they are useful enough to

arrive at some fruitful conclusions. Out of three main variants of phenomenological models, viz. kinematic, hydrodynamic and field-theoretic models¹², the present work deals with kinematical models where the dark energy representative Λ is assumed to be a function of time. Recently, Ray et al.¹³ and Mukhopadhyay et al.¹⁴ have shown the equivalence of four Λ models, viz. $\Lambda \sim (\dot{a}/a)^2$, $\Lambda \sim \ddot{a}/a$, $\Lambda \sim \rho$ and $\Lambda \sim \dot{H}$ for spatially flat ($k = 0$) Universe. But, since the closed ($k = 1$) and open ($k = -1$) Universes cannot be entirely ruled out, so there is enough reason to investigate dark energy for general k . In this work, therefore, one of the equivalent Λ models, viz. $\Lambda \sim (\dot{a}/a)^2$ is selected to solve Einstein equations for general k in order to have a broader view of accelerating Universe.

The scheme of the investigation is as follows: Sec. 2 and 3 deals respectively with the Field equations and their solutions while some physical feature arising out of this work are described in Sec. 4. Finally, some conclusions are made in Sec. 5.

2. Field Equations for the Spherically Symmetric Space-times

The Einstein field equations are given by

$$R^{ij} - \frac{1}{2}Rg^{ij} = -8\pi G \left[T^{ij} - \frac{\Lambda}{8\pi G}g^{ij} \right] \quad (1)$$

where the cosmological term Λ is time-dependent, i.e. $\Lambda = \Lambda(t)$.

Let us choose the spherically symmetric FLRW metric

$$ds^2 = -dt^2 + a(t)^2 \left[\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \right] \quad (2)$$

where k is the curvature constant and $a = a(t)$ is the scale factor. For the metric given by equation (2), the field equations (1) yield Friedmann and Raychaudhuri equations respectively given by

$$\left(\frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2} = \frac{8\pi G\rho}{3} + \frac{\Lambda}{3}, \quad (3)$$

$$\left(\frac{\ddot{a}}{a} \right) = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3} \quad (4)$$

where c , the velocity of light in vacuum, is assumed to be unity.

The generalized energy conservation law, when both Λ and G vary, is derived by Shapiro et al.¹⁵ using Renormalization Group Theory as well as by Vereschagin and Yegorian¹⁶ using Gurzadyan-Xue formula¹⁷. Since in the present work G is assumed as a constant and Λ is a variable, then the above mentioned generalized conservation law reduces to the particular form

$$8\pi G(p + \rho) \left(\frac{\dot{a}}{a} \right) = -\frac{8\pi G}{3}\dot{\rho} - \frac{\dot{\Lambda}}{3}. \quad (5)$$

The barotropic equation of state relating pressure and density is given by

$$p = \omega\rho \quad (6)$$

where the barotropic index ω can assume the values 0, 1/3, 1 and -1 for pressureless dust, electromagnetic radiation, stiff fluid and vacuum fluid respectively.

From (1), using equation (6), we get,

$$\rho = \frac{3}{4\pi G(1+3\omega)} \left(\frac{\Lambda}{3} - \frac{\ddot{a}}{3} \right) \quad (7)$$

Again, differentiating equation (3) and using equations (5) - (7) we obtain the differential equation

$$\left(\frac{\dot{a}}{a} \right)^2 + \frac{2}{1+3\omega} \left(\frac{\ddot{a}}{a} \right) + \frac{k}{a^2} = \left(\frac{1+\omega}{1+3\omega} \right) \Lambda. \quad (8)$$

3. Solutions for the Phenomenological Model $\Lambda = 3\alpha(\dot{a}/a)^2$

Using the *ansatz* $\Lambda = 3\alpha(\dot{a}/a)^2$, we immediately get from equation (8)

$$\frac{\ddot{a}}{\dot{a}} = [3\alpha(1+\omega) - (3\omega+1)] \frac{\dot{a}}{2a} - (3\omega+1) \frac{k}{2a\dot{a}}. \quad (9)$$

The above equation after simplification reduces to the form

$$a\dot{a} \frac{d}{dt} [\ln(\dot{a}a^{-s/2})] = -\frac{(3\omega+1)k}{2} \quad (10)$$

where $s = 3\alpha(1+\omega) - (3\omega+1)$.

Let us now study the following case when $s = -2$. In this case equation (10) reduces to

$$a\dot{a} \frac{d}{dt} [\ln(a\dot{a})] = -\frac{(3\omega+1)k}{2}. \quad (11)$$

Solving equation (11) we get our solution set as

$$a(t) = \left[C_0' t + C_1' - \frac{(3\omega+1)}{2} k t^2 \right]^{1/2}, \quad (12)$$

$$H(t) = \frac{C_0' - (1+3\omega)kt}{2 \left[C_0' t + C_1' - \frac{(3\omega+1)}{2} k t^2 \right]}, \quad (13)$$

$$\rho(t) = \frac{3(1-3\alpha)}{16\pi G} \frac{\left[C_0' - \frac{(3\omega+1)}{2} k t \right] + 2k \left[C_0' t + C_1' - \frac{(3\omega+1)}{2} k t^2 \right]}{\left[C_0' t + C_1' - \frac{(3\omega+1)}{2} k t^2 \right]^2}, \quad (14)$$

$$\Lambda(t) = \frac{3\alpha [C_0' - (3\omega+1)kt]^2}{4 \left[C_0' t + C_1' - \frac{(3\omega+1)}{2} k t^2 \right]^2} \quad (15)$$

where $C_0' = 2C_0$, $C_1' = 2C_1$, C_0 and C_1 being constants of integration.

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If we impose the boundary condition $a(t) = 0$ when $t = 0$, then $C_1 = 0$ which implies $C_1' = 0$. Then the simplified solution set becomes

$$a(t) = \left[C_0' t - \frac{(3\omega + 1)}{2} k t^2 \right]^{1/2}, \quad (16)$$

$$H(t) = \frac{C_0' - (1 + 3\omega)kt}{2 \left[C_0' t - \frac{(3\omega + 1)}{2} k t^2 \right]}, \quad (17)$$

$$\rho(t) = 3 \frac{[(\alpha + 1)C_0' - (1 + 3\omega)kt^2 + 2(1 + 3\omega)kC_0't - \frac{(1+3\omega)}{2}kt^2]}{16\pi G(1 + 3\omega)[C_0't - \frac{(1+3\omega)}{2}kt^2]^2}, \quad (18)$$

$$\Lambda(t) = \frac{3\alpha [C_0' - (3\omega + 1)kt]^2}{4 \left[C_0' t - \frac{(3\omega + 1)}{2} k t^2 \right]^2}. \quad (19)$$

It is clear from equation (19) that for a repulsive Λ , α must be positive whereas $\alpha = 0$ implies a null Λ . This means that we are getting Einstein's expanding Universe without Λ .

4. Physical Features of the Solutions

4.1. Density of the Universe Ω

The above solution set is obtained by assuming $s = -2$. Now, $s = -2$ means

$$\frac{2}{3(1 - \alpha)(1 + \omega)} = \frac{1}{2}. \quad (20)$$

For $k = 0$ we get from equations (16), (17), (19) and (20) respectively $a(t) \propto t^{2/3(1-\alpha)(1+\omega)}$, $H(t) \propto 1/t$, $\rho(t) \propto 1/t^2$ and $\Lambda(t) \propto 1/t^2$. These results were obtained by Ray et al.¹³ for flat ($k = 0$) Universe. Again, from equation (20) we have

$$\frac{(3\omega - 1)}{(1 + \omega)} = 3\alpha. \quad (21)$$

Since equation (19) suggests that for a repulsive Λ we must have $\alpha > 0$, then from equation (21) we find that either $\omega > 1/3$ or $\omega < -1$. For $\omega > 1/3$ we get a Universe where contribution of electromagnetic radiation is negligible (for radiation dominated Universe, $\omega = 1/3$) while $\omega < -1$ signifies the presence of phantom energy. Again, using equation (18), the expression for cosmic matter energy density Ω_m can be easily derived and is given by

$$\Omega_m = \frac{2(\alpha + 1)}{(1 + 3\omega)} + 4k \frac{[C_0't - \frac{(1+3\omega)}{2}kt^2]}{[C_0' - (1 + 3\omega)kt]^2}. \quad (22)$$

Also, from the *ansatz* $\Lambda = 3\alpha(\dot{a}/a)^2$ we get

$$\Omega_\Lambda = \alpha. \quad (23)$$

Then, using equations (21) - (23) we obtain

$$\Omega_m + \Omega_\Lambda = 1 + \frac{4k[C_0't - \frac{(1+3\omega)}{2}kt^2]}{[C_0' - (1+3\omega)kt]^2}. \quad (24)$$

For the flat Universe ($k = 0$), equation (24) reduces to the case of Ray et al.¹³. Again, equation (24) shows that at time $t = 0$, the sum of Ω_m and Ω_Λ becomes independent of the curvature constant k and takes a unit value whatever may be the value of k . On the other hand, when t tends to infinity, from (24) we have

$$\Omega_m + \Omega_\Lambda = 1 - \frac{2}{1+3\omega}. \quad (25)$$

From equation (25) we again observe that, $\Omega_m + \Omega_\Lambda$ is independent of k . Thus, both the early and late phases of the Universe exhibit the same behaviour so far as the curvature dependency of the sum of Ω_m and Ω_Λ is concerned. It has already been shown that for physically valid α , either $\omega > 1/3$ or $\omega < -1$. In the former case, $2/(1+3\omega) < 1$ and hence by equation (25)

$$0 < \Omega_m + \Omega_\Lambda < 1. \quad (26)$$

But, for $\omega < -1$ we have $-2/(1+3\omega) < 1$ and hence equation (25) provides the following constraint

$$1 < \Omega_m + \Omega_\Lambda < 2. \quad (27)$$

The above two relations (26) and (27) suggest that in distant future not only the sum total of matter and dark energy density will be independent of curvature of space but also they will be either less than (for $\omega > 1/3$) or greater than (for $\omega < -1$) unity which misfits with the present status of the sum of two type of energy densities dominating the present Universe. This result is very important for visualizing the cosmic evolution in future.

Now, let us suppose that the Universe is composed of a mixture of two types of fluids having barotropic indices ω_a and ω_b (say). Then, from equation (25) we get

$$(\Omega_m + \Omega_\Lambda)_a = 1 - \frac{2}{1+3\omega_a}, \quad (28)$$

$$(\Omega_m + \Omega_\Lambda)_b = 1 - \frac{2}{1+3\omega_b}. \quad (29)$$

If $(\Omega_m + \Omega_\Lambda)_{avg}$ be the average of $(\Omega_m + \Omega_\Lambda)_a$ and $(\Omega_m + \Omega_\Lambda)_b$ then

$$(\Omega_m + \Omega_\Lambda)_{avg} = 1 - \left[\frac{2+3(\omega_a+\omega_b)}{(1+3\omega_a)(1+3\omega_b)} \right]. \quad (30)$$

Equation (30) shows that when $\omega_a + \omega_b = -2/3$, then

$$(\Omega_m + \Omega_\Lambda)_{avg} = 1. \quad (31)$$

This means that like the early and present Universe, for late Universe also the sum of Ω_m and Ω_Λ will be unity only if the Universe contains a mixture of two types

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of fluids rather than a single fluid. Since $\omega_a + \omega_b = -2/3$ and it has already been shown that either $\omega > 1/3$ or $\omega < -1$, then let us suppose, $\omega_a = 1/3 + \epsilon$ and $\omega_b = -1 - \epsilon$ where $\epsilon > 0$. Therefore, for large value of t , when the Universe is filled with a mixture of two types of fluids (one of them being phantom-fluid) and if the value of barotropic index of one fluid is $-(1/3 + \epsilon)$ and that of the other is $(-1 - \epsilon)$, then the average value of the sum of Ω_m and Ω_Λ will be equal to one.

4.2. Deceleration parameter q

Let us now consider the expression for the deceleration parameter q which is given by

$$q = -\frac{a\ddot{a}}{\dot{a}^2} = -\left(1 + \frac{\dot{H}}{H^2}\right). \quad (32)$$

If the Universe is composed of two fluids with equation of state parameters ω_a and ω_b , then each of them will have some effect on the dynamics of the Universe. So for calculating the value of the deceleration parameter q , contributions coming from each component should be taken into account. If q_a and q_b be the values of two separate parts of q coming from fluids having barotropic indices ω_a and ω_b respectively, then using equation (17) we get from equation (33) the following two expressions

$$q_a = \frac{C_0'^2}{[C_0' - (2 + 3\epsilon)kt]^2}, \quad (33)$$

$$q_b = \frac{C_0'^2}{[C_0' + (2 + 3\epsilon)kt]^2}. \quad (34)$$

If q_{eff} be the effective value for q , coming after considering the separate parts q_a and q_b , then

$$q_{eff} = \frac{4C_0'^3 kt}{[C_0'^2 - (2 + 3\epsilon)^2 k^2 t^2]^2}. \quad (35)$$

Equation (35) shows that the sign of q depends only on two quantities, viz., the integration constant $C_0' (= 2C_0)$ and the curvature constant k . If for simplicity we assume C_0 to be positive then we get an accelerating or a decelerating Universe according as $k < 0$ or $k > 0$. This result can be interpreted as follows. The Universe is made of two types of fluids having equation of state parameters ω_a and ω_b , one of which is acting as a prohibitor and another as a supporter of cosmic acceleration. In the previous matter dominated phase, k had a small positive value (i.e. $q_a > q_b$) and as a result the Universe was decelerating. But at a certain time during cosmic evolution, the second type of fluid (viz., phantom fluid) took the upper-hand (i.e. $q_a < q_b$) and consequently k has become slightly negative. That is why the present Universe is in a state of acceleration. A very small positive or negative value of

the curvature constant do not contradict the observational result that the present Universe is nearly flat. Further, the change of sign of q shows that the cosmic acceleration is a recent phenomenon.

5. Conclusions

The present work, apart from being a generalization of an earlier one¹³, has revealed some new and interesting physical features also. For a flat Universe, all the results of Ray et al.¹³ can be recovered from the expressions of $a(t)$, $H(t)$, $\rho(t)$ and $\Lambda(t)$ of the present work. Moreover, it has been possible to trace the entire cosmic evolution, starting from the Big-Bang and extending to distant future. The most significant result is related to the cosmic matter and dark energy densities for non-flat Universe. It has been shown that for non-flat Universe, $\Omega_m + \Omega_\Lambda = 1$ only when the Universe is composed of two types of fluids, one with $\omega > 1/3$ and another with $\omega < -1$. This means that both stiff-fluid ($\omega = 1$) and phantom-fluid ($\omega = -1$) can be one of the two constituents of the Universe when the Universe is not flat. The evolution of the deceleration parameter q shows that a non-flat Universe would be decelerating in the past and accelerating at present. This result is very significant for Λ -CDM cosmology.

Another interesting point is the absence of Big-rip even in the presence of a fluid with $\omega < -1$. Caldwell¹⁸ and Caldwell et al.¹⁹ demonstrated the occurrence of a Big-rip in the presence of a fluid with supernegative equation of state. It may be mentioned here that Gonzalez-Diaz²⁰ has shown that by a proper generalization of the Chaplygin-gas model, a Big-rip can be avoided even in the presence of phantom energy whereas Abdalla et al.²¹ have arrived at the same result through a slight modification of GTR. But in the present work, a cosmic doomsday is shown to be impossible within the normal framework of GTR. One of the reasons of it may be the presence of another fluid apart from the phantom fluid. That other fluid may act as an inhibitor of Big-rip. It may be mentioned that Štefančič²² has developed a model in which the dark energy component and the matter component interact with each other resulting in the appearance of phantom energy out of non-phantom matter. The present work can be considered as a counter example of that because here phantom matter, in the presence of another component as a mediator, can behave as a non-phantom matter.

Finally, it is to be noted that the present work has demonstrated that although current observational data points towards a $k = 0$ Universe, yet we are not in a position right now to completely rule out $k = \pm 1$ cosmologies.

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